(1+1)-dimensional formalism and quasi-local conservation equations

Jong Hyuk Yoon
Department of Physics, Konkuk University,
Seoul 143-701, Korea
and
Enrico Fermi Institute, University of Chicago,
5640 S. Ellis Av., Chicago, IL 60637, U.S.A.
yoonjh@gr.uchicago.edu

A set of exact quasi-local conservation equations is obtained in the (1+1)-dimensional description of the Einstein's equations of (3+1)-dimensional spacetimes. These equations are interpreted as quasi-local energy, linear momentum, and angular momentum conservation equations. In the asymptotic region of asymptotically flat spacetimes, it is shown that these quasi-local conservation equations reduce to the conservation equations of Bondi energy, linear momentum, and angular momentum, respectively. When restricted to the quasi-local horizon of a generic spacetime, which is defined without referring to the infinity, the quasi-local conservation equations coincide with the conservation equations on the stretched horizon studied by Price and Thorne. All of these quasi-local quantities are expressed as invariant two-surface integrals, and geometrical interpretations in terms of the area of a given two-surface and a pair of null vector fields orthogonal to that surface are given.

I. INTRODUCTION AND KINEMATICS

For the past few decades, there has been enormous progress in general relativity since the pioneering works of the late Professor A. Lichnerowicz on mathematical relativity. His contributions to general relativity are diverse as well as profound, not least because he put the Einstein's equations on a firm mathematical foundation as partial differential equations and theories of connections[1]. Connections also plays important roles in Yang-Mills gauge theories, since gauge theories are nothing but theories of connections coupled to matter fields. Therefore, in this International Conference commemorating Professor A. Lichnerowicz, it seems appropriate to discuss a relatively unknown formalism of general relativity, which is based on the very idea of connections.

This note is about (1+1)-dimensional description of general relativity of (3+1)-dimensional spacetimes, treating the remaining 2-dimensional spatial dimensions as a fibre space. In this framework, all the notions in the theory of fibre bundles such as a fibre space, connections, and the structure group appear naturally. Instead of going into the details of the formalism itself[2, 3, 4], however, I will describe the key ideas briefly, mainly to fix the notations, and then quickly move on to discuss issues that are more immediate, namely, the problem of defining the quasi-local conservation equations using the (1+1)-dimensional formalism of (3+1)-dimensional spacetimes.

Let us begin by mentioning a few facts about quasi-local conservation equations. In general relativity there have been many attempts to obtain quasi-local conservation equations[4, 5, 6, 7, 8, 9, 10]. One of the motivations of these efforts is the expectation that quasi-local conservation equations allow us to predict certain aspects of a quasi-local region of a given spacetime without actually solving the Einstein's equations for that region. Recall that in the Newtonian theory, the conservation of total momentum immediately follows from Newton's third law,

$$\vec{F}_{\rm total} = \frac{d}{dt}(\sum_i \vec{p_i}) = 0, \tag{1}$$

which is no more than the consistency condition implementing Newton's second law. In general relativity, the consistency conditions for evolution are already incorporated into the Einstein's equations through the constraint equations, from which global conservation equations were found. In this note we will show that, from the Einstein's equations in the (1+1)-dimensional description, one can find conservation equations of a stronger form, namely, *quasi-local* conservation equations. These equations, which are integro-differential equations over a compact two-dimensional space, are naturally interpreted as quasi-local energy, linear momentum, and angular momentum conservation equations[11].

Let us consider the following line element

$$ds^2 = -2dudv - 2hdu^2 + {\rm e}^{\sigma}\rho_{ab}\left(dy^a + A_+^{\ a}du + A_-^{\ a}dv\right)\left(dy^b + A_+^{\ b}du + A_-^{\ b}dv\right), \tag{2}$$

where $+, -$ stands for u, v, respectively [3, 4, 12, 13, 14, 15, 16, 17]. To understand the geometry of this metric, it is convenient to introduce the following vector fields,

$$\hat{\partial}_+ := \partial_+ - A_+^{\ a}\partial_a, \tag{3}$$

$$\hat{\partial}_- := \partial_- - A_-^{\ a}\partial_a, \tag{4}$$

where we defined the following short-hand notations

$$\partial_+ := \frac{\partial}{\partial u}, \qquad \partial_- := \frac{\partial}{\partial v}, \qquad \partial_a := \frac{\partial}{\partial y^a} \ (a = 2, 3). \tag{5}$$

The inner products of the vector fields $\{\hat{\partial}_\pm, \partial_a\}$ are given by

$$\begin{aligned} &< \hat{\partial}_+,\ \hat{\partial}_+ >= -2h, \qquad < \hat{\partial}_+,\ \hat{\partial}_- >= -1, \qquad < \hat{\partial}_-,\ \hat{\partial}_- >= 0, \\ &< \hat{\partial}_\pm,\ \partial_a >= 0, \qquad < \partial_a,\ \partial_b >= \mathrm{e}^\sigma \rho_{ab}. \end{aligned} \tag{6}$$

The hypersurface u = constant is a null hypersurface generated by the out-going null vector field $\hat{\partial}_-$, which is orthogonal to the vector fields $\{\partial_a\}$. Notice that v is the *affine* parameter of the out-going null vector field. The hypersurface v = constant is generated by the vector field $\hat{\partial}_+$ whose norm is $-2h$, which can be either negative, zero, or positive. The intersection of two hypersurfaces u, v = constant defines a spacelike compact two-surface N_2, which are coordinatized by y^a. The metric on N_2 is decomposed into the area element e^σ and the conformal two-metric ρ_{ab}, which is normalized to have a unit determinant

$$\det \rho_{ab} = 1. \tag{7}$$

In the terminology of the fibre bundles, the base manifold is the (1+1)-dimensional space-time coordinatized by (u, v), and the fibre space is 2-dimension spacelike space N_2. The vector fields $\{\hat{\partial}_\pm\}$, which are orthogonal to $\{\partial_a\}$, is the horizontal vector field, and $\{\partial_a\}$ is tangent to the fibre space N_2. The fields $A_\pm^{\ a}$ are the corresponding connections valued in the diffeomorphisms of the two-surface N_2[2, 3, 4].

For later uses, we shall write down the future-directed in-going null vector field n and out-going null vector field l, orthogonal to two-surface N_2 at each spacetime point. They are given by

$$n := \hat{\partial}_+ - h\hat{\partial}_-, \tag{8}$$
$$l := \hat{\partial}_-, \tag{9}$$

and are normalized such that

$$< n,\ l >= -1. \tag{10}$$

If we further assume that $A_-^{\ a} = 0$, then the metric (2) becomes identical to the metric studied in [16]. In this note, however, we shall retain the $A_-^{\ a}$ field, since its presence will make the N_2-diffeomorphism invariant Yang-Mills type gauge theory aspect of this formalism transparent. Apart from the N_2-diffeomorphism invariance, there are other residual symmetries that preserve the metric (2), which are the reparametrization of u, and the transformation that shifts of the origin of the affine parameter v at each point of N_2[17].

The complete set of the vacuum Einstein's equations are found to be[15]

$$\begin{aligned} (a) \qquad & \mathrm{e}^\sigma D_+ D_- \sigma + \mathrm{e}^\sigma D_- D_+ \sigma + 2\mathrm{e}^\sigma (D_+\sigma)(D_-\sigma) - 2\mathrm{e}^\sigma (D_- h)(D_-\sigma) - \frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab} F_{+-}^{\ \ a} F_{+-}^{\ \ b} \\ & + \mathrm{e}^\sigma R_2 - h\mathrm{e}^\sigma \Big\{ (D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) \Big\} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} (b) \qquad & -\mathrm{e}^\sigma D_+^2 \sigma - \frac{1}{2}\mathrm{e}^\sigma (D_+\sigma)^2 - \mathrm{e}^\sigma (D_- h)(D_+\sigma) + \mathrm{e}^\sigma (D_+ h)(D_-\sigma) + 2h\mathrm{e}^\sigma (D_- h)(D_-\sigma) \\ & + \mathrm{e}^\sigma F_{+-}^{\ \ a}\partial_a h - \frac{1}{4}\mathrm{e}^\sigma \rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \partial_a\Big(\rho^{ab}\partial_b h\Big) \\ & + h\Big\{ -\mathrm{e}^\sigma (D_+\sigma)(D_-\sigma) + \frac{1}{2}\mathrm{e}^\sigma \rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) + \frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}^{\ \ a}F_{+-}^{\ \ b} - \mathrm{e}^\sigma R_2 \Big\} \\ & + h^2 \mathrm{e}^\sigma \Big\{ (D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) \Big\} = 0, \end{aligned} \tag{12}$$

$$(c) \qquad 2\mathrm{e}^\sigma (D_-^2 \sigma) + \mathrm{e}^\sigma (D_-\sigma)^2 + \frac{1}{2}\mathrm{e}^\sigma \rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) = 0, \tag{13}$$

$$(d) \qquad D_-\Big(\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}^{\ \ b}\Big) - \mathrm{e}^\sigma \partial_a (D_-\sigma) - \frac{1}{2}\mathrm{e}^\sigma \rho^{bc}\rho^{de}(D_-\rho_{bd})(\partial_a\rho_{ce}) + \partial_b\Big(\mathrm{e}^\sigma \rho^{bc} D_-\rho_{ac}\Big) = 0, \tag{14}$$

$$(e) \qquad -D_+\Big(\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}^{\ \ b}\Big) - \mathrm{e}^\sigma \partial_a (D_+\sigma) - \frac{1}{2}\mathrm{e}^\sigma \rho^{bc}\rho^{de}(D_+\rho_{bd})(\partial_a\rho_{ce}) + \partial_b\Big(\mathrm{e}^\sigma \rho^{bc} D_+\rho_{ac}\Big)$$

$$+2h\mathrm{e}^\sigma\partial_a(D_-\sigma)+h\mathrm{e}^\sigma\rho^{bc}\rho^{de}(D_-\rho_{bd})(\partial_a\rho_{ce})+2\mathrm{e}^\sigma\partial_a(D_-h)-2\partial_b\Big(h\mathrm{e}^\sigma\rho^{bc}D_-\rho_{ac}\Big)=0, \tag{15}$$

$$(f)\qquad \begin{aligned}&-2\mathrm{e}^\sigma D_-^2h-2\mathrm{e}^\sigma(D_-h)(D_-\sigma)+\mathrm{e}^\sigma D_+D_-\sigma+\mathrm{e}^\sigma D_-D_+\sigma+\mathrm{e}^\sigma(D_+\sigma)(D_-\sigma)\\&+\frac{1}{2}\mathrm{e}^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd})+\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^aF_{+-}{}^b-2h\mathrm{e}^\sigma\Big\{D_-^2\sigma+\frac{1}{2}(D_-\sigma)^2\\&+\frac{1}{4}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\Big\}=0,\end{aligned} \tag{16}$$

$$(g)\qquad \begin{aligned}&h\Big\{\mathrm{e}^\sigma D_-^2\rho_{ab}-\mathrm{e}^\sigma\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})+\mathrm{e}^\sigma(D_-\rho_{ab})(D_-\sigma)\Big\}\\&-\frac{1}{2}\mathrm{e}^\sigma\Big(D_+D_-\rho_{ab}+D_-D_+\rho_{ab}\Big)+\frac{1}{2}\mathrm{e}^\sigma\rho^{cd}\Big\{(D_-\rho_{ac})(D_+\rho_{bd})+(D_-\rho_{bc})(D_+\rho_{ad})\Big\}\\&-\frac{1}{2}\mathrm{e}^\sigma\Big\{(D_-\rho_{ab})(D_+\sigma)+(D_+\rho_{ab})(D_-\sigma)\Big\}\\&+\mathrm{e}^\sigma(D_-\rho_{ab})(D_-h)+\frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ac}\rho_{bd}F_{+-}{}^cF_{+-}{}^d-\frac{1}{4}\mathrm{e}^{2\sigma}\rho_{ab}\rho_{cd}F_{+-}{}^cF_{+-}{}^d=0.\end{aligned} \tag{17}$$

Here R_2 is the scalar curvature of N_2, and we defined the diffN_2-covariant derivatives as follows,

$$F_{+-}{}^a:=\partial_+A_-^a-\partial_-A_+^a-[A_+,A_-]^a_{\mathrm{L}}, \tag{18}$$

$$D_\pm\sigma:=\partial_\pm\sigma-[A_\pm,\sigma]_{\mathrm{L}}, \tag{19}$$

$$D_\pm h:=\partial_\pm h-[A_\pm,h]_{\mathrm{L}}, \tag{20}$$

$$D_\pm\rho_{ab}:=\partial_\pm\rho_{ab}-[A_\pm,\rho]_{\mathrm{L}ab}. \tag{21}$$

The bracket $[A_\pm,f]_{\mathrm{L}ab\cdots}$ is the Lie derivative of $f_{ab\cdots}$ along the vector field $A_\pm:=A_\pm^a\partial_a$, defined as

$$[A_\pm,f]_{\mathrm{L}ab\cdots}:=A_\pm^c\partial_cf_{ab\cdots}+f_{cb\cdots}\partial_aA_\pm^c+f_{ac\cdots}\partial_bA_\pm^c-w(\partial_cA_\pm^c)f_{ab\cdots}, \tag{22}$$

where w is the weight of the tensor density $f_{ab\cdots}$. One can also compute the scalar curvature R of the metric (2) and integrate it over spacetime. It is given by

$$\begin{aligned}I_0&=\int du\,dv\,d^2y\,\mathrm{e}^\sigma R\\&=\int du\,dv\,d^2y\,L_0+\text{surface integrals},\end{aligned} \tag{23}$$

where the "Lagrangian" function L_0 is given by[3]

$$\begin{aligned}L_0=&-\frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^aF_{+-}{}^b+\mathrm{e}^\sigma(D_+\sigma)(D_-\sigma)-\frac{1}{2}\mathrm{e}^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd})-\mathrm{e}^\sigma R_2\\&-2\mathrm{e}^\sigma(D_-h)(D_-\sigma)-h\mathrm{e}^\sigma(D_-\sigma)^2+\frac{1}{2}h\mathrm{e}^\sigma\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}).\end{aligned} \tag{24}$$

One can easily recognize that this "Lagrangian" function L_0 is in a form of a (1+1)-dimensional field theory Lagrangian. In geometrical terms the function L_0 describes how the (1+1)-dimensionsal spacetime and 2-dimensional fibre space are imbedded into an enveloping (3+1)-dimensional spacetime. Each term in (24) is manifestly diffN_2-invariant, and the y^a-dependence of each term is completely "hidden" in the Lie derivatives. In this sense we may regard the fibre space N_2 as a kind of "internal" space as in a Yang-Mills theory, with the infinite dimensional group of diffeomorphism of N_2 as the Yang-Mills gauge symmetry. Thus, the above function L_0 is describable as a (1+1)-dimensional Yang-Mills type gauge theory interacting with (1+1)-dimensional scalar fields and non-linear sigma fields of generic types.

II. A SET OF QUASI-LOCAL CONSERVATION EQUATIONS

Notice that the four equations (11), (12) and (15) are partial differential equations that are *first-order* in D_- derivatives. Therefore it is of particular interest to study these four equations, since they are close analogues to the Einstein's constraint equations in the usual (3+1) formalism. Thus, in this formalism, the *natural* vector field

that defines the evolution is D_-. Then the momenta $\pi_I = \{\pi_h, \pi_\sigma, \pi_a, \pi^{ab}\}$ conjugate to the configuration variables $q^I = \{h, \sigma, A_+^{\ a}, \rho_{ab}\}$ are *defined* as

$$\pi_I := \frac{\partial L_0}{\partial(D_- q^I)}. \tag{25}$$

They are found to be

$$\pi_h = -2\mathrm{e}^\sigma(D_-\sigma), \tag{26}$$
$$\pi_\sigma = -2\mathrm{e}^\sigma(D_- h) - 2h\mathrm{e}^\sigma(D_-\sigma) + \mathrm{e}^\sigma(D_+\sigma), \tag{27}$$
$$\pi_a = \mathrm{e}^{2\sigma}\rho_{ab}F_{+-}^{\ \ \ b}, \tag{28}$$
$$\pi^{ab} = h\mathrm{e}^\sigma\rho^{ac}\rho^{bd}(D_-\rho_{cd}) - \frac{1}{2}\mathrm{e}^\sigma\rho^{ac}\rho^{bd}(D_+\rho_{cd}). \tag{29}$$

Notice that π^{ab} is traceless

$$\pi^a_{\ a} = 0, \tag{30}$$

due to the identities that are direct consequences of the condition (7),

$$\rho^{ab}D_\pm\rho_{ab} = 0. \tag{31}$$

The "Hamiltonian" function H_0 *defined* as

$$H_0 := \pi_I D_- q^I - L_0 \tag{32}$$

is found to be

$$H_0 = H + \text{total divergences}, \tag{33}$$

where H is given by

$$\begin{aligned} H = & -\frac{1}{2}\mathrm{e}^{-\sigma}\pi_\sigma\pi_h + \frac{1}{4}h\mathrm{e}^{-\sigma}\pi_h^2 - \frac{1}{2}\mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b + \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ac}\rho_{bd}\pi^{ab}\pi^{cd} \\ & + \frac{1}{2}\pi_h(D_+\sigma) + \frac{1}{2h}\pi^{ab}(D_+\rho_{ab}) + \frac{1}{8h}\mathrm{e}^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \mathrm{e}^\sigma R_2. \end{aligned} \tag{34}$$

In terms of these canonical variables $\{\pi_I, q^I\}$, the first-order equations (11), (12), and (15) can be written as, after a little algebra,

$$\begin{aligned} \text{(i)} \qquad & \pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma - hD_+\pi_h - \partial_+\Big(h\,\pi_h + 2\mathrm{e}^\sigma D_+\sigma\Big) \\ & +\partial_a\Big(h\,\pi_h A_+^{\ a} + 2A_+^{\ a}\mathrm{e}^\sigma D_+\sigma + 2h\mathrm{e}^{-\sigma}\rho^{ab}\pi_b + 2\rho^{ab}\partial_b h\Big) = 0, \end{aligned} \tag{35}$$

$$\text{(ii)} \qquad H - \partial_+\pi_h + \partial_a\Big(A_+^{\ a}\pi_h + \mathrm{e}^{-\sigma}\rho^{ab}\pi_b\Big) = 0, \tag{36}$$

$$\begin{aligned} \text{(iii)} \qquad & \partial_+\pi_a - \partial_b(A_+^{\ b}\pi_a) - \pi_b\partial_a A_+^{\ b} - \pi_\sigma\partial_a\sigma + \partial_a\pi_\sigma - \pi_h\partial_a h - \pi^{bc}\partial_a\rho_{bc} \\ & +\partial_b(\pi^{bc}\rho_{ac}) + \partial_c(\pi^{bc}\rho_{ab}) - \partial_a(\pi^{bc}\rho_{bc}) = 0. \end{aligned} \tag{37}$$

These are the four first-order equations in the gauge (2), and it is these equations that we are concerned with in this note. Notice that the equations (35) and (36) are *divergence*-type equations. If we contract the equations (37) by an arbitrary function ξ^a of $\{v, y^b\}$ such that

$$\partial_+\xi^a = 0, \tag{38}$$

then the resulting equation is also a divergence-type equation,

$$\pi^{ab}\pounds_\xi\rho_{ab} + \pi_\sigma\pounds_\xi\sigma + \pi_h\pounds_\xi h + \pi_a\pounds_\xi A_+^{\ a} - \partial_+(\xi^a\pi_a) + \partial_a\Big(-\xi^a\pi_\sigma + 2\pi^{ab}\xi^c\rho_{bc} + A_+^{\ a}\xi^b\pi_b\Big) = 0, \tag{39}$$

where $\pounds_\xi$ is the Lie derivative along the vector field $\xi := \xi^a\partial_a$.

The integrals of these equations over a compact two-surface N_2 become, after the normalization by $1/16\pi$,

$$\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma - hD_+\pi_h\right), \tag{40}$$

$$\frac{\partial}{\partial u}P(u,v) = \frac{1}{16\pi}\oint d^2y\, H, \tag{41}$$

$$\frac{\partial}{\partial u}L(u,v;\xi) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\pounds_\xi\rho_{ab} + \pi_\sigma\pounds_\xi\sigma - h\pounds_\xi\pi_h - A_+^{\ a}\pounds_\xi\pi_a\right) \qquad (\partial_+\xi^a = 0), \tag{42}$$

where in the last integral we used the fact that

$$\oint d^2y\,\pounds_\xi f = \oint d^2y\,\partial_a(\xi^a f) = 0 \tag{43}$$

for a scalar density f with the weight -1. Here $U(u,v)$, $P(u,v)$, and $L(u,v;\xi)$ are invariant two-surface integrals defined as

$$U(u,v) := \frac{1}{16\pi}\oint d^2y\left(h\,\pi_h + 2\mathrm{e}^\sigma D_+\sigma\right) + \bar{U}, \tag{44}$$

$$P(u,v) := \frac{1}{16\pi}\oint d^2y\,(\pi_h) + \bar{P}, \tag{45}$$

$$L(u,v;\xi) := \frac{1}{16\pi}\oint d^2y\,(\xi^a\pi_a) + \bar{L} \qquad (\partial_+\xi^a = 0), \tag{46}$$

where $\bar{U}$, $\bar{P}$, and $\bar{L}$ are undetermined subtraction terms. Notice that these subraction terms must be u-independent,

$$\frac{\partial\bar{U}}{\partial u} = \frac{\partial\bar{P}}{\partial u} = \frac{\partial\bar{L}}{\partial u} = 0, \tag{47}$$

in order to satisfy the equations (40), (41), and (42), respectively. In general the subtraction terms are not unique, and the "right" subtraction term may not even exist at all in a generic situation. One natural criterion for the "right" choice of subtraction term would be that it must be chosen such that the quasi-local physical quantities reproduce "standard" values in the well-known limiting cases.

One can write the r.h.s. of the equation (40) in a more symmetric and suggestive form as follows. To do this, let us contract the equation (37) with $A_+^{\ a}$ and integrate over N_2 to obtain the following equation

$$\oint d^2y\left(A_+^{\ a}\partial_+\pi_a\right) = \oint d^2y\left(\pi^{ab}\pounds_{A_+}\rho_{ab} + \pi_\sigma\pounds_{A_+}\sigma - h\pounds_{A_+}\pi_h\right). \tag{48}$$

If we use the definition of diffN_2-covariant derivatives $D_\pm$ and the equation (48), then the equation (40) can be written as

$$\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\partial_+\rho_{ab} + \pi_\sigma\partial_+\sigma - h\partial_+\pi_h - A_+^{\ a}\partial_+\pi_a\right), \tag{49}$$

where the integrand on the r.h.s. assumes the canonical form of energy-flux, which is typically given by

$$T_{0+} \sim \sum_i \pi_i\partial_+\phi^i, \tag{50}$$

where ϕ^i is a generic field and π_i is its conjugate momentum. Notice that the r.h.s. of the conservation equations (42) and (49) match exactly, if we interchange the derivatives in the integrands

$$\pounds_\xi \longleftrightarrow \partial_+. \tag{51}$$

In a region of a spacetime where $\partial/\partial u$ is timelike, these quasi-local equations becomes quasi-local conservation equations, which relate the instantaneous rates of changes of two-surface integrals at a given u-time to the associated net flux integrals. Let us remark that, unlike the Tamburino-Winicour's quasi-local conservation equations[11] which are "weak" conservation equations since the Ricci flat conditions (i.e. the full vacuum Einstein's equations) were assumed in their derivation, our quasi-local conservation equations are "strong" conservation equations since only the four first-order equations were used in the derivation.

It is interesting to notice that we can obtain yet another quasi-local conservation equation. This is simply achieved by writing the equation (48) as

$$\frac{\partial}{\partial u}\oint d^2y\,(A_+^{\ a}\pi_a) = \oint d^2y\left(\pi^{ab}\pounds_{A_+}\rho_{ab} + \pi_\sigma\pounds_{A_+}\sigma - h\pounds_{A_+}\pi_h + \pi_a\partial_+ A_+^{\ a}\right), \tag{52}$$

which relates the instantaneous u-derivative of the two-surface integral on the l.h.s. to the net flux integral on the right. However, the r.h.s. of this equation is not quite "canonical" due to the last term. If we restrict the field $A_+^{\ a}$ such that it satisfies the u-independent condition

$$\partial_+ A_+^{\ a} = 0, \tag{53}$$

which is essentially the same condition (38) that ξ^a satisfies, then the last term in the r.h.s. of the equation (52) drops out, and we obtain the following equation

$$\frac{\partial}{\partial u}J(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\pounds_{A_+}\rho_{ab} + \pi_\sigma\pounds_{A_+}\sigma - h\pounds_{A_+}\pi_h\right). \tag{54}$$

Here $J(u,v)$ is defined as

$$J(u,v) := \frac{1}{16\pi}\oint d^2y\,(A_+^{\ a}\pi_a) + \bar{J} \qquad (\partial_+ A_+^{\ a} = 0), \tag{55}$$

where $\bar{J}$ is an undetermined subtraction term. The r.h.s. of the equation (54) now represents a flux of the canonical form

$$\sum_i \pi_i \pounds_{A_+}\phi^i, \tag{56}$$

just as the r.h.s. of the equations (42) and (49) do.

III. GEOMETRICAL INTERPRETATIONS

Remarkably, the two-surface integrals (44), (45), (46), and (55), which were derived using the metric (2), can be expressed geometrically, in terms of the area of the two-surface and a pair of in-going and out-going null vector fields orthogonal to that surface. In order to show this, we need to invoke the definitions of in-going and out-going null vector fields $\{n,\ l\}$ defined in the section I.

A. Quasi-local energy

Let us first observe that the integral in (44) can be written as the Lie derivative of the scalar density e^σ along the in-going null vector field n,

$$\begin{aligned}\oint d^2y\left(h\,\pi_h + 2\mathrm{e}^\sigma D_+\sigma\right) &= 2\oint d^2y\,\mathrm{e}^\sigma\left(D_+\sigma - hD_-\sigma\right)\\ &= 2\oint d^2y\,\pounds_n \mathrm{e}^\sigma.\end{aligned} \tag{57}$$

One finds that

$$\oint d^2y\,\pounds_n\mathrm{e}^\sigma = \pounds_n\mathcal{A}, \tag{58}$$

where $\mathcal{A}$ is the area of N_2,

$$\mathcal{A} = \oint d^2y\,\mathrm{e}^\sigma. \tag{59}$$

The identity (58) follows trivially from the observation that the null vector field n is out of (in fact, orthogonal to) the two-surface, which means that the order of the integration over d^2y and the Lie derivative $\pounds_n$ in (57) is interchangeable. Thus we have

$$\frac{1}{16\pi}\oint d^2y\left(h\,\pi_h + 2\mathrm{e}^{\sigma}D_+\sigma\right) = \frac{1}{8\pi}\pounds_n\mathcal{A}. \tag{60}$$

For a reference term $\bar{U}$, let us choose

$$\bar{U} := -\frac{1}{8\pi}\pounds_{\bar{n}}\mathcal{A}, \tag{61}$$

where $\bar{n}$ is a future-directed in-going null vector field of a background reference spacetime $d\bar{s}^2$ into which the two-surface N_2 (with the same metric $\mathrm{e}^{\sigma}\rho_{ab}$) is embedded,

$$d\bar{s}^2 = -2dudv - 2\bar{h}du^2 + \mathrm{e}^{\sigma}\rho_{ab}\left(dy^a + \bar{A}_+{}^a du + \bar{A}_-{}^a dv\right)\left(dy^b + \bar{A}_+{}^b du + \bar{A}_-{}^b dv\right). \tag{62}$$

Notice that $\bar{n}$, which is given by

$$\bar{n} := \Big(\frac{\partial}{\partial u} - \bar{A}_+{}^a\frac{\partial}{\partial y^a}\Big) - \bar{h}\Big(\frac{\partial}{\partial v} - \bar{A}_-{}^a\frac{\partial}{\partial y^a}\Big), \tag{63}$$

is a function of $\bar{h}$ and $\bar{A}_\pm{}^a$ only, the embedding degrees of freedom of the two-surface. Thus, the quasi-local energy of a given two-surface N_2 is defined relative to some fixed background reference spacetime, and is zero when the two-surface under consideration is embedded into the fixed background reference spacetime, i.e. when

$$n = \bar{n}. \tag{64}$$

Therefore, the quantity $U(u,v)$, which will be interpreted as the quasi-local energy, becomes

$$U(u,v) := \frac{1}{8\pi}\pounds_n\mathcal{A} - \frac{1}{8\pi}\pounds_{\bar{n}}\mathcal{A}. \tag{65}$$

It is given by the rate of change of the area of a given two-surface along the future-directed *in*-going null vector field n, relative to some background null vector field $\bar{n}$. Notice that this definition is entirely geometrical, referring to the area of a given two-surface and the orthogonal null vector fields n (and $\bar{n}$) only.

B. Quasi-local linear momentum

The two-surface integral (45) can be also written geometrically in a similar way. It becomes

$$\frac{1}{16\pi}\oint d^2y\,(\pi_h) = -\frac{1}{8\pi}\oint d^2y\,\mathrm{e}^{\sigma}D_-\sigma = -\frac{1}{8\pi}\oint d^2y\,\mathrm{e}^{\sigma}\pounds_l\sigma = -\frac{1}{8\pi}\pounds_l\mathcal{A}. \tag{66}$$

Therefore, if we choose the reference term $\bar{P}$ as

$$\bar{P} := \frac{1}{8\pi}\pounds_{\bar{l}}\mathcal{A}, \tag{67}$$

where $\bar{l}$ is a future-directed out-going null vector field of a background reference spacetime, then P becomes,

$$P(u,v) = -\frac{1}{8\pi}\pounds_l\mathcal{A} + \frac{1}{8\pi}\pounds_{\bar{l}}\mathcal{A}. \tag{68}$$

Thus, the quasi-local integral $P(u,v)$, which is to be interpreted as the quasi-local linear momentum, is given by the rate of change of the area of a given two-surface along the future-directed *out*-going null vector field l relative to $\bar{l}$.

C. Quasi-local angular momentum

Let us also write down the two-surface integral (46) in a geometrical way. Notice that the Lie bracket of the two null vector fields $\{n, l\}$ is given by

$$[n,\ l]_{\rm L} = -F_{+-}{}^{a}\partial_a + (D_- h)l. \tag{69}$$

Thus, (46) becomes

$$\begin{aligned} \frac{1}{16\pi}\oint d^2y\,(\xi^a\pi_a) &= \frac{1}{16\pi}\oint d^2y\,{\rm e}^{\sigma}\xi_a F_{+-}{}^{a} \\ &= -\frac{1}{16\pi}\oint d^2y\,{\rm e}^{\sigma}\xi_a[n,\ l]^a_{\rm L}, \end{aligned} \tag{70}$$

where

$$\xi_a := \phi_{ab}\xi^b = {\rm e}^{\sigma}\rho_{ab}\xi^b. \tag{71}$$

This shows that (70) is an invariant two-surface integral of the Lie-bracket $[n,\ l]_{\rm L}$ projected to the spacelike vector field $\xi := \xi^a\partial_a$. If we choose the reference term $\bar{L}$ as

$$\bar{L} := \frac{1}{16\pi}\oint d^2y\,{\rm e}^{\sigma}\xi_a[\bar{n},\ \bar{l}]^a_{\rm L}, \tag{72}$$

then (46) becomes

$$L(u,v;\xi) = -\frac{1}{16\pi}\oint d^2y\,{\rm e}^{\sigma}\xi_a[n,\ l]^a_{\rm L} + \frac{1}{16\pi}\oint d^2y\,{\rm e}^{\sigma}\xi_a[\bar{n},\ \bar{l}]^a_{\rm L} \qquad (\partial_+\xi^a = 0). \tag{73}$$

It must be stressed ξ^a is an arbitrary function of $\{v, y^b\}$. In particular, it need not satisfy any Killing's equations associated with isometries of a spacetime, or of a two-surface within a given spacetime. The quantity $L(u,v;\xi)$, a linear functional of ξ^a, can be interpreted as the quasi-local angular momentum of a two-surface associated with an arbitrary function ξ^a (of y^b), as we will see later.

D. Quasi-local Carter's constant

Likewise, the fourth integral (55) can be written as

$$J(u,v) = -\frac{1}{16\pi}\oint d^2y\,{\rm e}^{2\sigma}\rho_{ab}A_+{}^{a}[n,\ l]^b_{\rm L} + \frac{1}{16\pi}\oint d^2y\,{\rm e}^{2\sigma}\rho_{ab}A_+{}^{a}[\bar{n},\ \bar{l}]^b_{\rm L} \qquad (\partial_+ A_+{}^{a} = 0), \tag{74}$$

which may be interpreted as a quasi-local, finite, analog of the Carter's "fourth" constant, as we shall see in the next section.

IV. ASYMPTOTICALLY FLAT LIMITS

The equations (40), (41), and (42) turn out to be quasi-local energy, linear momentum, and angular momentum conservation equations, respectively[4], and the equation (54) is interpreted as a quasi-local conservation equation of the generalized Carter's constant[18, 19, 20]. In this section we shall evaluate the two-surface integrals (65), (68), (73) and the associated flux integrals in the limiting asymptotically flat region where $N_2 = S_2$, and show that they all reduce to the well-known Bondi energy, linear momentum, angular momentum, and the corresponding flux integrals defined at the null infinity. The asymptotic form of the "fourth" integral (74) at the null infinity will be also computed, and it will be shown that it is proportional to the total angular momentum squared.

In the limit where the affine parameter v approaches to infinity, the asymptotic form of the Kerr metric becomes

$$\begin{aligned} ds^2 \quad &\longrightarrow -2dudv - \Big(1 - \frac{2m}{v} + \cdots\Big)du^2 + \Big(\frac{4ma\sin^2\vartheta}{v} - \frac{4ma^3\sin^2\vartheta\cos^2\vartheta}{v^3} + \cdots\Big)dud\varphi \\ &+v^2\Big(1 + \frac{a^2\cos^2\vartheta}{v^2} + \cdots\Big)d\vartheta^2 + v^2\sin^2\vartheta\Big(1 + \frac{a^2}{v^2} + \cdots\Big)d\varphi^2 \\ &+\sin^2\vartheta\Big(\frac{4ma^3}{v^3} + \frac{8m^2a^3}{v^4} + \cdots\Big)dvd\varphi - \Big(\frac{a^2\sin^2\vartheta}{v^2} + \cdots\Big)dv^2, \end{aligned} \tag{75}$$

where $\partial/\partial u$ is asymptotic to the timelike Killing vector field at infinity. The asymptotic fall-off rates of the metric coefficients can be read off from the above metric[21, 22, 23, 24, 25],

$$\mathrm{e}^{\sigma} = v^2(\sin\vartheta)\Big\{1+O(\frac{1}{v^2})\Big\}, \tag{76}$$

$$\rho_{\vartheta\vartheta} = \Big(\frac{1}{\sin\vartheta}\Big)\Big\{1+\frac{C(u,\vartheta,\varphi)}{v}+O(\frac{1}{v^2})\Big\}, \tag{77}$$

$$\rho_{\varphi\varphi} = (\sin\vartheta)\Big\{1-\frac{C(u,\vartheta,\varphi)}{v}+O(\frac{1}{v^2})\Big\}, \tag{78}$$

$$\rho_{\vartheta\varphi} = O(\frac{1}{v^2}), \tag{79}$$

$$2h = 1-\frac{2m}{v}+O(\frac{1}{v^2}), \tag{80}$$

$$A_{+}{}^{\varphi} = \frac{2ma}{v^3}+O(\frac{1}{v^4}), \tag{81}$$

$$A_{-}{}^{\varphi} = \frac{2ma^3}{v^5}+O(\frac{1}{v^6}), \tag{82}$$

$$A_{\pm}{}^{\vartheta} = O(\frac{1}{v^6}). \tag{83}$$

From these asymptotic behaviors, we can deduce the fall-off rates of the following derivatives,

$$\begin{aligned}
&\partial_{+}\sigma = O(\frac{1}{v^2}), \quad \partial_{-}\sigma = \frac{2}{v}+O(\frac{1}{v^2}), \quad \partial_{+}\rho_{ab} = O(\frac{1}{v}), \quad \partial_{-}\rho_{ab} = O(\frac{1}{v^2}), \quad \pounds_{\xi}\rho_{ab} = O(\frac{1}{v}),\\
&\pi_h = -4v\sin\vartheta+O(1), \quad \pi_\sigma = -2v\sin\vartheta+O(1), \quad \pi^{ab} = -\frac{1}{2}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(\partial_{+}\rho_{cd})+O(1),\\
&\pi_\varphi = 6ma\sin^3\vartheta+O(\frac{1}{v}), \quad \pi_\vartheta = O(\frac{1}{v^2}).
\end{aligned} \tag{84}$$

A. The Bondi energy-loss relation

Since the integrand of the r.h.s. of (49) assumes the typical form of energy-flux, we expect that it represents the energy-flux carried by gravitational radiation crossing S_2. Then the l.h.s. of (49) should be the instantaneous rate of change in the gravitational energy of the region enclosed by S_2. The energy-flux integral in general does not have a definite sign, since it includes the energy-flux carried by the in-coming as well as the out-going gravitational radiation. But in the asymptotically flat region, the energy-flux integral turns out to be negative-definite, representing the physical situation that there is no in-coming flux coming from the infinity.

Let us now show that the equation (49) reduces to the Bondi energy-loss formula[17] in the asymptotic region of asymptotically flat spacetimes. To show this, let us first calculate $U(u,v)$ in the limit $v \longrightarrow \infty$. Since the null vector fields n and l asymptotically approach to

$$\begin{aligned}
&n \longrightarrow \partial_{+} - \Big(\frac{1}{2}-\frac{m}{v}\Big)\partial_{-},\\
&l \longrightarrow \partial_{-},
\end{aligned} \tag{85}$$

the natural background spacetime is the flat spacetime so that the embedding degrees of freedom are given by

$$\bar{A}_{\pm}{}^{a} = 0, \qquad 2\bar{h} = 1. \tag{86}$$

That is, the background fields $\bar{n}$ and $\bar{l}$ become

$$\bar{n} = \partial_{+} - \frac{1}{2}\partial_{-}, \qquad \bar{l} = \partial_{-}. \tag{87}$$

Then it follows trivially that the total energy at the null infinity coincides with the Bondi energy $U_{\mathrm{B}}(u)$,

$$\lim_{v\rightarrow\infty} U(u,v) := U_{\mathrm{B}}(u) = m, \tag{88}$$

where m is the Bondi mass of asymptotically flat spacetimes. One can further show that the equation (49) is just the Bondi energy-loss formula,

$$\frac{d}{du}U_{\rm B}(u) = -\lim_{v\to\infty}\frac{1}{32\pi}\oint_{S_2} d\Omega\; v^2\rho^{ab}\rho^{cd}(\partial_+\rho_{ac})(\partial_+\rho_{bd}), \tag{89}$$

or, equivalently,

$$\frac{d}{du}U_{\rm B}(u) = -\frac{1}{16\pi}\oint_{S_2} d\Omega\;(\partial_+ C)^2 \le 0, \tag{90}$$

where we used the expressions (77) and (78). Notice that the negative-definite energy-flux is a bilinear of the traceless *current* $j^a_{\ b}$ defined as

$$j^a_{\ b} := \rho^{ac}\partial_+\rho_{bc} \qquad (j^a_{\ a} = 0), \tag{91}$$

representing the shear degrees of freedom of gravitational radiation.

B. The Bondi linear momentum and linear momentum-flux

Let us now evaluate $P(u,v)$ in (68) and the corresponding quasi-local momentum flux integral in the asymptotic region of asymptotically flat spacetimes. We find that the total linear momentum $P(u,v)$ becomes zero in the asymptotic limit,

$$\lim_{v\to\infty} P(u,v) := P_{\rm B}(u) = 0, \tag{92}$$

from which we infer that the total momentum flux is zero,

$$\frac{d}{du}P_{\rm B}(u) = 0. \tag{93}$$

The result (93) can be also obtained by evaluating each term in the "Hamiltonian" function H (41). To evaluate the momentum-flux term by term, let us notice that the fourth and the seventh term in (34), which are non-zero individually, add up to zero asymptotically,

$$\begin{aligned}\frac{1}{2h}{\rm e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{8h}{\rm e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \;&=\; \frac{h}{2}{\rm e}^{\sigma}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) \\ &\longrightarrow\; O\Big(\frac{1}{v^2}\Big),\end{aligned} \tag{94}$$

where we used the definition (29) of π^{ab}. All other non-vanishing terms are given by

$$\lim_{v\to\infty}\frac{1}{16\pi}\oint_{S_2} d^2y\left(\frac{1}{4}h{\rm e}^{-\sigma}\pi_h^2\right) = \frac{1}{2}, \tag{95}$$

$$\lim_{v\to\infty}\frac{1}{16\pi}\oint_{S_2} d^2y\left(\frac{1}{2}{\rm e}^{-\sigma}\pi_h\pi_\sigma\right) = 1, \tag{96}$$

$$\lim_{v\to\infty}\frac{1}{16\pi}\oint_{S_2} d^2y\,{\rm e}^{\sigma}R_2 = \frac{1}{4}\chi, \tag{97}$$

where $\chi = 2$ for a two-sphere S_2. Therefore we have

$$\frac{d}{du}P_{\rm B}(u) = 0. \tag{98}$$

C. The Bondi angular momentum and angular momentum-flux

The total angular momentum at the null infinity is naturally defined as the limiting value of the general quasi-local angular momentum $L(u,v;\xi)$ in (73),

$$\lim_{v\to\infty} L(u,v;\xi) := L_{\rm B}(u;\xi). \tag{99}$$

Since the background fields $\bar{n}$ and $\bar{l}$ in (87) commute,

$$[\bar{n},\ \bar{l}]_{\rm L} = 0, \tag{100}$$

it follows that

$$\bar{L}_{\rm B}(u;\xi) = 0 \tag{101}$$

for all ξ. Let ξ be asymptotic to the azimuthal Killing vector field of the Kerr spacetime such that

$$\xi := \xi^a \partial_a \longrightarrow \frac{\partial}{\partial \varphi}. \tag{102}$$

Then, we have

$$\begin{aligned} L_{\rm B}(u;\xi) &= \frac{1}{16\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, (6ma) \sin^3\vartheta \\ &= ma, \end{aligned} \tag{103}$$

which is just the total angular momentum of the Kerr spacetime.

The total angular momentum flux at the instant u is given by the asymptotically limiting form of the the equation (42), which is

$$\frac{dL_{\rm B}}{du} := \lim_{v\to\infty} \frac{1}{16\pi} \oint_{S_2} d^2y \left(\pi^{ab} \pounds_\xi \rho_{ab} + \pi_\sigma \pounds_\xi \sigma - h \pounds_\xi \pi_h - A_+{}^a \pounds_\xi \pi_a\right). \tag{104}$$

Let us evaluate each term in the r.h.s. of this equation. The first term is given by

$$\begin{aligned} \pi^{ab} \pounds_\xi \rho_{ab} &= \Big\{ -\frac{1}{2} {\rm e}^\sigma \rho^{ac} \rho^{bd} (\partial_+ \rho_{cd}) + O(1) \Big\} \pounds_\xi \rho_{ab} \\ &= -\sin\vartheta (\partial_+ C)(\pounds_\xi C) + O(\frac{1}{v}), \end{aligned} \tag{105}$$

so that we have

$$\oint_{S_2} d^2y\, \pi^{ab} \pounds_\xi \rho_{ab} \longrightarrow - \oint_{S_2} d\Omega\ (D_+ C)(\pounds_\xi C). \tag{106}$$

The second term becomes

$$\pi_\sigma \pounds_\xi \sigma = \Big\{ -2v \sin\vartheta + O(1) \Big\} \pounds_\xi \sigma. \tag{107}$$

Since σ is asymptotically given by

$$\sigma = 2 \ln v + \ln |\sin\vartheta| + \ln \Big\{ 1 + O\Big(\frac{1}{v^2}\Big)\Big\}, \tag{108}$$

we have

$$\pounds_\xi \sigma = O\Big(\frac{1}{v^2}\Big), \tag{109}$$

so that the second term becomes

$$\oint_{S_2} d^2y\, \pi_\sigma \pounds_\xi \sigma = O\Big(\frac{1}{v}\Big) \longrightarrow 0. \tag{110}$$

The third term becomes

$$\begin{aligned} h \pounds_\xi \pi_h &= -\pi_h \pounds_\xi h + \pounds_\xi (h \pi_h) \\ &= \pounds_\xi (4m \sin\vartheta + h\pi_h) + O\Big(\frac{1}{v}\Big), \end{aligned} \tag{111}$$

where we used that

$$\pounds_\xi(\sin\vartheta) = 0. \tag{112}$$

Thus we have

$$\oint_{S_2} d^2y\, h \pounds_\xi \pi_h = O\Big(\frac{1}{v}\Big) \longrightarrow 0. \tag{113}$$

The fourth term is of the order of

$$A_+^{\ a} \pounds_\xi \pi_a = O\Big(\frac{1}{v^3}\Big), \tag{114}$$

so that

$$\oint_{S_2} d^2y\, A_+^{\ a} \pounds_\xi \pi_a = O(\frac{1}{v^3}) \longrightarrow 0. \tag{115}$$

If we put together (106), (110), (113), and (115) into (104), then the total angular momentum flux at the null infinity is given by

$$\frac{dL_{\rm B}}{du} = -\lim_{v\to\infty} \frac{1}{32\pi} \oint_{S_2} d^2y\, {\rm e}^\sigma \rho^{ac} \rho^{bd} (\partial_+ \rho_{cd})(\pounds_\xi \rho_{ab}), \tag{116}$$

or, equivalently,

$$\frac{dL_{\rm B}}{du} = -\frac{1}{16\pi} \oint_{S_2} d\Omega\ (\partial_+ C)(\pounds_\xi C), \tag{117}$$

which is precisely the Bondi angular momentum flux at the null infinity[26].

D. Gravitational Carter's constant

Let us define the asymptotic quantity $J_{\rm B}(u)$ as

$$\lim_{v\to\infty} v^3 J(u,v) := J_{\rm B}(u). \tag{118}$$

Because of the equation (100), the subtraction term $\bar{J}_{\rm B}(u)$ is zero,

$$\bar{J}_{\rm B}(u) = 0. \tag{119}$$

Then the asymptotic integral $J_{\rm B}(u)$ becomes

$$J_{\rm B}(u) = 2(ma)^2. \tag{120}$$

Thus, $J_{\rm B}(u)$ is (twice of) the angular momentum squared, deserveing the name the *gravitational* analog of the Carter's "fourth" constant at the null infinity.

V. IN-GOING NULL COORDINATES

One might be also interested in applying this formalism to black holes, and try to obtain quasi-local quantities defined on the black hole horizon and corresponding fluxes incident on that horizon. For this problem, it is appropriate to choose a coordinate system adapted to the *in*-going null geodesics. Such a coordinate system is described by the metric

$$ds^2 = +2dudv - 2hdu^2 + {\rm e}^\sigma \rho_{ab} \left(dy^a + A_+^{\ a} du + A_-^{\ a} dv\right)\left(dy^b + A_+^{\ b} du + A_-^{\ b} dv\right). \tag{121}$$

In this coordinate system, the future-directed out-going and in-going null vector fields n' and l' are given by

$$n' := \hat{\partial}_+ + h\hat{\partial}_-, \tag{122}$$
$$l' := -\hat{\partial}_-, \tag{123}$$

respectively, which are normalized so that

$$< n',\ l' >= -1. \tag{124}$$

The inverse relation is given by

$$\hat{\partial}_+ = n' + hl', \tag{125}$$
$$\hat{\partial}_- = -l'. \tag{126}$$

If we repeat the same analysis as in the previous chapters using the metric (121), we obtain a new (but equivalent, of course) set of quasi-local conservation equations, which we present without derivations. The conjugate momenta are found to be

$$\pi_h = -2\mathrm{e}^{\sigma} D_-\sigma, \tag{127}$$
$$\pi_\sigma = -2\mathrm{e}^{\sigma} D_- h - 2h\mathrm{e}^{\sigma} D_-\sigma - \mathrm{e}^{\sigma} D_+\sigma, \tag{128}$$
$$\pi_a = \mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{b}, \tag{129}$$
$$\pi^{ab} = h\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}D_-\rho_{cd} + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}D_+\rho_{cd}, \tag{130}$$

and the "Hamiltonian" function H is given by

$$\begin{aligned} H = &-\frac{1}{2}\mathrm{e}^{-\sigma}\pi_h\pi_\sigma + \frac{1}{4}h\mathrm{e}^{-\sigma}\pi_h^2 - \frac{1}{2}\mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b + \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} \\ &-\frac{1}{2}\pi_h D_+\sigma - \frac{1}{2h}\pi^{ab}D_+\rho_{ab} + \frac{1}{8h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \mathrm{e}^{\sigma}R_2. \end{aligned} \tag{131}$$

A new set of quasi-local conservations equations are found to be

$$\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\partial_+\rho_{ab} + \pi_\sigma\partial_+\sigma - h\partial_+\pi_h - A_+{}^{a}\partial_+\pi_a\right), \tag{132}$$
$$\frac{\partial}{\partial u}P(u,v) = -\frac{1}{16\pi}\oint d^2y\, H, \tag{133}$$
$$\frac{\partial}{\partial u}L(u,v;\xi) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\pounds_\xi\rho_{ab} + \pi_\sigma\pounds_\xi\sigma - h\pounds_\xi\pi_h - A_+{}^{a}\pounds_\xi\pi_a\right) \qquad (\partial_+\xi^a = 0), \tag{134}$$
$$\frac{\partial}{\partial u}J(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}\pounds_{A_+}\rho_{ab} + \pi_\sigma\pounds_{A_+}\sigma - h\pounds_{A_+}\pi_h\right) \qquad (\partial_+A_+{}^{a} = 0), \tag{135}$$

where $U(u,v)$, $P(u,v)$, $L(u,v;\xi)$, and $J(u,v)$ are defined as

$$U(u,v) := \frac{1}{16\pi}\oint d^2y\,(h\,\pi_h - 2\mathrm{e}^{\sigma}D_+\sigma) + \bar{U}, \tag{136}$$
$$P(u,v) := \frac{1}{16\pi}\oint d^2y\,(\pi_h) + \bar{P}, \tag{137}$$
$$L(u,v;\xi) := \frac{1}{16\pi}\oint d^2y\,(\xi^a\pi_a) + \bar{L}, \tag{138}$$
$$J(u,v) := \oint d^2y\,(A_+{}^{a}\pi_a) + \bar{J}, \tag{139}$$

where $\bar{U}$, $\bar{P}$, $\bar{L}$, and $\bar{J}$ are undetermined reference terms as before. Notice that we could also have written the equation (132) as

$$\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y\left(\pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma - hD_+\pi_h\right), \tag{140}$$

as in (40). In geometrical terms, these quasi-local quantities can be expressed as,

$$
\begin{aligned}
U(u,v) &:= -\frac{1}{8\pi}\pounds_{n'}\mathcal{A} + \frac{1}{8\pi}\pounds_{\bar{n}'}\mathcal{A}, \\
P(u,v) &= \frac{1}{8\pi}\pounds_{l'}\mathcal{A} - \frac{1}{8\pi}\pounds_{\bar{l}'}\mathcal{A}, \\
L(u,v;\xi) &= \frac{1}{16\pi}\oint d^2y\, \mathrm{e}^{\sigma}\xi_a[n,\ l]^a_{\mathrm{L}} - \frac{1}{16\pi}\oint d^2y\, \mathrm{e}^{\sigma}\xi_a[\bar{n}',\ \bar{l}']^a_{\mathrm{L}} \qquad (\partial_+\xi^a = 0), \\
J(u,v) &= \frac{1}{16\pi}\oint d^2y\, \mathrm{e}^{2\sigma}\rho_{ab}A_+^{\ a}[n,\ l]^b_{\mathrm{L}} - \frac{1}{16\pi}\oint d^2y\, \mathrm{e}^{2\sigma}\rho_{ab}A_+^{\ a}[\bar{n}',\ \bar{l}']^b_{\mathrm{L}} \qquad (\partial_+ A_+^{\ a} = 0).
\end{aligned} \tag{141}
$$

Here $\bar{n}'$, $\bar{l}'$ are future-directed in-going and out-going null vector fields of a background reference spacetime

$$
d\bar{s}^2 = +2dudv - 2\bar{h}du^2 + \mathrm{e}^{\sigma}\rho_{ab}\left(dy^a + \bar{A}_+^{\ a}du + \bar{A}_-^{\ a}dv\right)\left(dy^b + \bar{A}_+^{\ b}du + \bar{A}_-^{\ b}dv\right), \tag{142}
$$

such that

$$
\begin{aligned}
\bar{n}' &:= \Big(\frac{\partial}{\partial u} - \bar{A}_+^{\ a}\frac{\partial}{\partial y^a}\Big) + \bar{h}\Big(\frac{\partial}{\partial v} - \bar{A}_-^{\ a}\frac{\partial}{\partial y^a}\Big), \\
\bar{l}' &:= -\Big(\frac{\partial}{\partial v} - \bar{A}_-^{\ a}\frac{\partial}{\partial y^a}\Big).
\end{aligned} \tag{143}
$$

VI. QUASI-LOCAL HORIZON

Recall that the event horizon is a global concept which is inseparable from the notion of infinity. Therefore, in order to discuss the dynamics of black holes quasi-locally, we have to introduce a new notion of *quasi-local horizon*, which refers to the quasi-local region only. We define the quasi-local horizon $\mathcal{H}$ as a three-dimensional hypersurface on which the vector field $\hat{\partial}_+$, which is an arbitrary vector field except that it is asymptotic to the timelike Killing vector at the infinity, has a zero norm,

$$
<\hat{\partial}_+, \hat{\partial}_+> = -2h = 0. \tag{144}
$$

The location of the quasi-local horizon $\mathcal{H}$ can be found by solving the equation

$$
h(u,v,y^a) = 0 \tag{145}
$$

for v, and the generator of the quasi-local horizon is given by

$$
\hat{\partial}_+ = \frac{\partial}{\partial u} - A_+^{\ a}\frac{\partial}{\partial y^a}, \tag{146}
$$

which is out-going null on $\mathcal{H}$.

Let us remark a few properties of this quasi-local horizon. First, the quasi-local horizon is defined for generic spacetimes that do not have isometries in general, and its location is not fixed, but *varies* as much as the choice of the vector field $\hat{\partial}_+$ does. In this sense the quasi-local horizon is not a spacetime invariant but a *covariant* notion. Even the signature of the vector field $\hat{\partial}_+$ is not determined *a priori*. In the region where $\hat{\partial}_+$ is non-spacelike, however, the quasi-local horizon may be regarded as a generalization of the Killing horizon, since it is defined as the hypersurface where $\hat{\partial}_+$ has a zero norm. For instance, for the Schwarzschild solution, we have

$$
2h = 1 - \frac{2m}{v}, \tag{147}
$$

so that $h = 0$ on the Killing horizon $v = 2m$. Moreover, since the quasi-local horizon is generated by the out-going null vector fields, all the fluxes crossing the quasi-local horizon is purely *in-going*, just like the stretched horizon of Price and Thorne.

In this section, we shall delimit our discussions of the quasi-local conservation equations to the quasi-local horizon $\mathcal{H}$, and find that the quasi-local conservation equations on $\mathcal{H}$ coincides *exactly* with the quasi-local conservation equations of Price and Thorne[27] defined on the *stretched* horizon.

A. Surface gravity κ

In order to discuss the dynamics of quasi-local horizon, it is useful to introduce the notion of the surface gravity κ to the generic, time-dependent, quasi-local horizon. On the quasi-local horizon $\mathcal{H}$ on which $h = 0$, the vector field χ defined as

$$\chi := \frac{\partial}{\partial u} - A_+^{\ a} \frac{\partial}{\partial y^a} \tag{148}$$

becomes a generator of $\mathcal{H}$, since we have

$$\chi \cdot \chi = -2h|_{\mathcal{H}} = 0. \tag{149}$$

Hence $\nabla_A(\chi \cdot \chi)|_{\mathcal{H}}$ is normal to $\mathcal{H}$, which means that there exists a function κ defined on $\mathcal{H}$ such that

$$\nabla_A(\chi \cdot \chi)|_{\mathcal{H}} := -2\kappa \chi_A|_{\mathcal{H}}, \tag{150}$$

where A is a spacetime index such that $A = \{+, -, a\}$. Notice that this function κ can be defined on any null hypersurface. When the null hypersurface coincides with the event horizon, this function is the *surface gravity* of the corresponding black hole. But one may use the same terminology for κ on a quasi-local horizon $\mathcal{H}$, since it is a (segment of) null hypersurface beyond which a local observer whose worldline $\hat{\partial}_+$ has a zero norm on $\mathcal{H}$ does not have access to. It may be instructive to notice that, for the Kerr black hole, χ is the generator of the event horizon, since it becomes

$$\chi \longrightarrow \frac{\partial}{\partial u} - \Omega_{\mathcal{H}} \frac{\partial}{\partial \varphi}, \tag{151}$$

where $\partial/\partial u$ and $\partial/\partial \varphi$ are timelike and axial Killing vector fields, and $\Omega_{\mathcal{H}}$ is the angular velocity of the Kerr horizon relative to the Killing time.

Let us compute κ on the quasi-local horiozn $\mathcal{H}$. In the basis $\{\partial/\partial u,\ \partial/\partial v,\ \partial/\partial y^a\}$, the components of χ are given by

$$\begin{aligned} \chi^A &= (1,\ 0,\ -A_+^{\ a}), \\ \chi_A &:= g_{AB}\chi^B = (-2h,\ 1,\ 0). \end{aligned} \tag{152}$$

If we put (152) into (150), then we find that

$$\partial_+ h|_{\mathcal{H}} = \partial_a h|_{\mathcal{H}} = 0, \tag{153}$$

and that κ is given by

$$\kappa = D_- h|_{\mathcal{H}}. \tag{154}$$

Notice that, in general, κ is *not* constant over $\mathcal{H}$ so that

$$\partial_+ \kappa \neq 0, \qquad \partial_a \kappa \neq 0, \tag{155}$$

which reflects the dynamical nature of the quasi-local horizon $\mathcal{H}$.

B. Quasi-local energy conservation on $\mathcal{H}$

Notice that if we restrict the energy equation (140) to the quasi-local horizon $\mathcal{H}$, then it becomes,

$$\frac{\partial}{\partial u} U_{\mathcal{H}} = \frac{1}{16\pi} \oint_{\mathcal{H}} d^2 y \left\{ \frac{1}{2} \mathrm{e}^{\sigma} \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_+ \rho_{bd}) - \mathrm{e}^{\sigma} (D_+ \sigma)^2 - 2\mathrm{e}^{\sigma} \kappa D_+ \sigma \right\}, \tag{156}$$

$$U_{\mathcal{H}} := -\frac{1}{8\pi} \oint_{\mathcal{H}} d^2 y \, \mathrm{e}^{\sigma} D_+ \sigma + \bar{U}_{\mathcal{H}}. \tag{157}$$

This equation is identical to the integral of the following equation

$$\frac{\partial}{\partial u} \Sigma_{\mathcal{H}} + \theta_{\mathcal{H}} \Sigma_{\mathcal{H}} = -\frac{1}{8\pi} \kappa \, \theta_{\mathcal{H}} - \frac{1}{16\pi} \theta_{\mathcal{H}}^2 + \frac{1}{8\pi} \sigma_{ab}^{\mathcal{H}} \sigma_{\mathcal{H}}^{ab} \tag{158}$$

over the stretched horizon of Price and Thorne, where the notations are such that

$$
\begin{aligned}
&\Sigma_{\mathcal{H}} = -\frac{1}{8\pi}\theta_{\mathcal{H}}, \\
&\theta_{\mathcal{H}} = D_+\sigma, \\
&\sigma^{\mathcal{H}}_{ab} = \frac{1}{2}\mathrm{e}^{\sigma} D_+\rho_{ab}, \\
&\sigma^{ab}_{\mathcal{H}} = \phi^{ac}\phi^{bd}\sigma^{\mathcal{H}}_{cd} = \frac{1}{2}\mathrm{e}^{-\sigma}\rho^{ac}\rho^{bd}D_+\rho_{cd}, \\
&\kappa = D_- h|_{\mathcal{H}}.
\end{aligned} \tag{159}
$$

This equation was studied in detail in Eq. (6.112,E) in [27].

It is interesting to discuss the limiting case when the quasi-local horizon $\mathcal{H}$ coincides with the event horizon. When this happens, the area $A_{\mathcal{H}}$ of $\mathcal{H}$ always increases due to the area theorem, so that we have

$$
\frac{dA_{\mathcal{H}}}{du} = \oint_{\mathcal{H}} d^2y \, (\mathrm{e}^{\sigma} D_+\sigma) \geq 0. \tag{160}
$$

Furthermore, if the subtraction term $\bar{U}_{\mathcal{H}}$ is chosen zero, then by the equation (157), $U_{\mathcal{H}}$ is non-positive, and when the black hole no longer expands so that $D_+\sigma|_{\mathcal{H}} = 0$, then $U_{\mathcal{H}}$ becomes zero. For instance, for a Schwarzschild or Kerr black hole[27], we have

$$
U_{\mathcal{H}} = 0 \quad \text{if} \quad \bar{U}_{\mathcal{H}} \equiv 0. \tag{161}
$$

This counter-intuitive aspect is a manifestation of the well-known teleological nature of the event horizon. That is, when the event horizon evolves, its quasi-local energy must be negative so as to cancel out the positive in-flux of energy carried by subsequently in-falling matter or gravitational radiation, leaving $U_{\mathcal{H}} = 0$ when the black hole reaches the final stationary state.

C. Quasi-local momentum conservation equation on $\mathcal{H}$

Let us mention that the momentum equation (133) has a similar structure to the integrated Navier-Stokes equation for a viscous fluid[28],

$$
\frac{\partial P_i}{\partial u} = -\oint dS^k \Big(p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} \Big), \tag{162}
$$

where P_i and σ'_{ik} are the total momentum and the viscous term,

$$
P_i = \int dV \, (\rho v_i), \tag{163}
$$

$$
\sigma'_{ik} = \eta \Big(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x^l} \Big) + \zeta \delta_{ik}\frac{\partial v_l}{\partial x^l}, \tag{164}
$$

and η and ζ are the coefficients of shear and bulk viscosity, respectively. This equation tells us that the rate of the net momentum change of a fluid within a given volume is determined by the net momentum-flux across the two-surface enclosing the volume. Notice that the "Hamiltonian" function H in (131), which is at most quadratic in the conjugate momenta π_I, assumes the form of momentum-flux of a viscous fluid. Namely, terms quadratic in π_I may be viewed as responsible for direct momentum transfer, terms linear in π_I as viscosity terms, and terms independent of π_I as pressure terms. From this point of view, one may interpret the "Hamiltonian" function H as the gravitational momentum-flux and the two-surface integral

$$
P_{\mathcal{H}} = \frac{1}{16\pi}\oint d^2y \, (\pi_h) + \bar{P} \tag{165}
$$

as the quasi-local gravitational momentum within N_2. On the quasi-local horizon $\mathcal{H}$, the equation (133) becomes,

$$
\frac{\partial}{\partial u}P_{\mathcal{H}} = -\frac{1}{16\pi}\oint_{\mathcal{H}} d^2y \Big(\mathrm{e}^{\sigma}R_2 - \frac{1}{2}\mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b - \frac{1}{2}\mathrm{e}^{-\sigma}\pi_\sigma\pi_h - \frac{1}{2}\pi_h D_+\sigma \Big). \tag{166}
$$

The first term on the r.h.s. is given by the Euler number χ,

$$\frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\, \mathrm{e}^{\sigma} R_2 = \frac{1}{4}\chi, \tag{167}$$

where $\chi = 2$ for a two-sphere. The second and third terms are quadratic in the momenta, and the last term is linear in the momentum. It is curious that this conservation equation is missing in the work of Price and Thorne[27]. Let us note that, in terms of the configuration variables, the integrand of the r.h.s. of (166) can be written as

$$H_{\mathcal{H}} = \mathrm{e}^{\sigma} R_2 - \frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}F_{+-}{}^{b} - 2\mathrm{e}^{\sigma}\kappa D_{-}\sigma. \tag{168}$$

D. Quasi-local angular momentum conservation equation on $\mathcal{H}$

In this section, we shall show that the equation (134) when restricted to the surface $\mathcal{H}$ coincides with the quasi-local angular momentum equation of Price and Thorne on the stretched horizon. Let us first notice that the equation (134) on $\mathcal{H}$ can be written as

$$\frac{\partial}{\partial u}L_{\mathcal{H}} = \frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\Big\{ - \mathrm{e}^{\sigma}(2\kappa + D_{+}\sigma)\pounds_{\xi}\sigma + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_{+}\rho_{ab})(\pounds_{\xi}\rho_{cd}) - A_{+}^{\ a}\pounds_{\xi}\pi_{a}\Big\}, \tag{169}$$

$$L_{\mathcal{H}} := \frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\,(\xi^{a}\pi_{a}) + \bar{L}_{\mathcal{H}}. \tag{170}$$

The equation (169) turns out to be identical to the angular momentum conservation equation of Price and Thorne, which is given by

$$D_{+}\Pi^{\mathcal{H}}_{a} + \theta_{\mathcal{H}}\Pi^{\mathcal{H}}_{a} = -\frac{1}{8\pi}\ \kappa_{,a} - \frac{1}{16\pi}\theta_{\mathcal{H},a} + \frac{1}{8\pi}\sigma_{\mathcal{H}a}{}^{b}{}_{;b}, \tag{171}$$

$$\Pi^{\mathcal{H}}_{a} := -\frac{1}{16\pi}\mathrm{e}^{\sigma}\rho_{ab}F_{+-}{}^{b}. \tag{172}$$

To show this, let us write the equations (169) and (170) as

$$\frac{\partial}{\partial u}L_{\mathcal{H}} = \frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\Big\{2\mathrm{e}^{\sigma}\pounds_{\xi}\kappa + \mathrm{e}^{\sigma}\pounds_{\xi}(D_{+}\sigma) + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{ab}\pounds_{\xi}(D_{+}\rho_{ab}) + \mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}\pounds_{\xi}A_{+}^{\ b}\Big\}, \tag{173}$$

$$L_{\mathcal{H}} := \frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\,(\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}\xi^{b}) + \bar{L}_{\mathcal{H}}, \tag{174}$$

where we used the identity

$$\oint_{\mathcal{H}} d^2y\,\pounds_{\xi}f = \oint_{\mathcal{H}} d^2y\,\partial_{a}(\xi^{a}f) = 0 \tag{175}$$

for any scalar density f with the weight -1. Using the definitions of $L_{\mathcal{H}}$ in (174) and $\Pi^{\mathcal{H}}_{a}$ in (172), we obtain the following identity,

$$\begin{aligned}
&\frac{\partial}{\partial u}L_{\mathcal{H}} - \frac{1}{16\pi}\oint_{\mathcal{H}} d^2y\,(\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}\pounds_{\xi}A_{+}^{\ b})\\
&= -\oint_{\mathcal{H}} d^2y\Big\{\mathrm{e}^{\sigma}\xi^{a}(D_{+}\sigma)\Pi^{\mathcal{H}}_{a} + \mathrm{e}^{\sigma}(D_{+}\xi^{a})\Pi^{\mathcal{H}}_{a} + \mathrm{e}^{\sigma}\xi^{a}D_{+}\Pi^{\mathcal{H}}_{a} + \mathrm{e}^{\sigma}\Pi^{\mathcal{H}}_{a}[A_{+},\ \xi]^{a}_{\mathrm{L}}\Big\}\\
&= -\oint_{\mathcal{H}} d^2y\Big\{\mathrm{e}^{\sigma}\xi^{a}D_{+}\Pi^{\mathcal{H}}_{a} + \mathrm{e}^{\sigma}\xi^{a}(D_{+}\sigma)\Pi^{\mathcal{H}}_{a} + \mathrm{e}^{\sigma}\Pi^{\mathcal{H}}_{a}\Big(D_{+}\xi^{a} + [A_{+},\ \xi]^{a}_{\mathrm{L}}\Big)\Big\}\\
&= -\oint_{\mathcal{H}} d^2y\,\mathrm{e}^{\sigma}\xi^{a}(D_{+}\Pi^{\mathcal{H}}_{a} + \theta_{\mathcal{H}}\Pi^{\mathcal{H}}_{a}),
\end{aligned} \tag{176}$$

where we used the diffN_2-covariant derivative of ξ^{a},

$$D_{+}\xi^{a} := \partial_{+}\xi^{a} - [A_{+},\ \xi]^{a}_{\mathrm{L}} \tag{177}$$

defined in the section I, and the condition that

$$\partial_+ \xi^a = 0. \tag{178}$$

The first and second term on the r.h.s. of (173) become,

$$\frac{1}{16\pi}\oint_{\mathcal{H}} d^2y \left\{ 2\mathrm{e}^\sigma \pounds_\xi \kappa + \mathrm{e}^\sigma \pounds_\xi (D_+\sigma) \right\} = \oint_{\mathcal{H}} d^2y\, \mathrm{e}^\sigma \xi^a \Big(\frac{1}{8\pi}\kappa_{,a} + \frac{1}{16\pi}\theta_{\mathcal{H},a} \Big), \tag{179}$$

and the third term on the r.h.s. of (173) is given by

$$\begin{aligned}
\frac{1}{2}\mathrm{e}^\sigma \rho^{ab} \pounds_\xi (D_+\rho_{ab}) &= \frac{1}{2}\mathrm{e}^\sigma \rho^{ab} \Big\{ \xi^c \nabla_c (D_+\rho_{ab}) + (D_+\rho_{cb})(\nabla_a \xi^c) + (D_+\rho_{ac})(\nabla_b \xi^c) - (\nabla_c \xi^c)(D_+\rho_{ab}) \Big\} \\
&= \frac{1}{2}\mathrm{e}^\sigma \xi^c \Big\{ \nabla_c (\rho^{ab} D_+\rho_{ab}) - (\nabla_c \rho^{ab}) D_+\rho_{ab} \Big\} + \mathrm{e}^\sigma \rho^{ab} (D_+\rho_{bc})(\nabla_a \xi^c) \\
&= \mathrm{e}^\sigma \rho^{ab} (D_+\rho_{bc})(\nabla_a \xi^c) \\
&= -\xi^a \mathrm{e}^\sigma \rho^{bc} \nabla_b (D_+\rho_{ac}) + \mathrm{e}^\sigma \rho^{ab} \nabla_a (\xi^c D_+\rho_{bc}) \\
&= -2\xi^a \mathrm{e}^\sigma \sigma_{\mathcal{H}a}{}^b{}_{;b} + \nabla_a (\xi^c \mathrm{e}^\sigma \rho^{ab} D_+\rho_{bc}).
\end{aligned} \tag{180}$$

Here we used the unimodular condition (31), and the metricity condition

$$\nabla_a \sigma = \nabla_a \rho_{bc} = 0. \tag{181}$$

Notice that the shear tensor $\sigma_{\mathcal{H}a}{}^b$ is given by

$$\sigma_{\mathcal{H}a}{}^b := \phi^{bc}\sigma_{\mathcal{H}ac} = \frac{1}{2}\rho^{bc} D_+\rho_{ac}. \tag{182}$$

Therefore we have

$$\oint_{\mathcal{H}} d^2y\, \frac{1}{2}\mathrm{e}^\sigma \rho^{ab} \pounds_\xi (D_+\rho_{ab}) = -\oint_{\mathcal{H}} d^2y\, 2\xi^a \mathrm{e}^\sigma \sigma_{\mathcal{H}a}{}^b{}_{;b}. \tag{183}$$

If we put together (176), (179), and (183), then (173) becomes

$$\oint_{\mathcal{H}} d^2y\, \mathrm{e}^\sigma \xi^a \Big\{ D_+ \Pi^{\mathcal{H}}_a + \theta_{\mathcal{H}} \Pi^{\mathcal{H}}_a + \frac{1}{8\pi}\kappa_{,a} + \frac{1}{16\pi}\theta_{\mathcal{H},a} - \frac{1}{8\pi}\sigma_{\mathcal{H}a}{}^b{}_{;b} \Big\} = 0, \tag{184}$$

for an arbitrary function ξ^a that satisfies (178). This shows that the two equations (169) and (171) are identical.

VII. SUMMARY

In this note, I presented a set of quasi-local conservation equations that were found while studying the Einstein's equations using the (1+1)-dimensional description, and studied physical significances of these conservation equations. The key observation is that the affine coordinate v can be treated as a natural time coordinate in this formalism. The subsequent Hamiltonian formalism was developed with respect to this time coordinate. One of the outcomes of this analysis is the discovery of a set of Bondi-like quasi-local conservation equations for the vacuum general relativity, which reproduce *both* the well-known conservation equations in the asymptotic null infinity in asymptotically flat spacetimes *and* the corresponding conservation equations on the inner quasi-local horizon of a generic dynamical spacetime. All of these quasi-local quantities are expressed in geometrically invariant terms such as the area of the two-surface and a pair of null vector fields orthogonal to that surface. It was also found that each quantity has a natural interpretation as the quasi-local energy, linear momentum, and angular momentum of a two-surface and corresponding fluxes crossing that surface.

In addition to the above quasi-local quantities, we also obtained the quasi-local analog of the Carter's "fourth" constant of *gravitational* field, which is somewhat like the angular momentum squared, measuring the "intrinsic" angular momentum of a two-surface. The Carter's "fourth" constant is known to exist for spacetimes that possess two commmuting Killing vector fields such as the Kerr black hole. But in our analysis, it was found that the quasi-local analog of Carter's constant exists under the condition

$$\partial_+ A_+{}^a = 0, \tag{185}$$

which is much less restrictive than the existence of two-commuting Killing fields.

Dynamics of the quasi-local horizon was discussed briefly, but it deserves further studies. Applications of this formalism to astrophysical problems involving black holes and gravitational radiations are extremely challenging. These problems are left for future works.

Acknowledgments

This work was supported by Konkuk University research grant in 2002.

[1] A. Lichnerowicz, *Théorie globale des connexions et des groupes d'holonomie* (Edizioni Cremonese 1955).
[2] Y.M. Cho, Q.H. Park, K.S. Soh and J.H. Yoon, Phys. Lett. B 286 (1992) 251.
[3] J.H. Yoon, Phys. Lett. B 308 (1993) 240.
[4] J.H. Yoon, Phys. Lett. A 292 (2001) 166.
[5] J. D. Brown and J.W. York, Phys. Rev. D 47 (1993) 1407.
[6] S. Hayward, Phys. Rev. D 49 (1994) 831.
[7] J. Nester, Class. Quantum Grav. 8 (1991) L19.
[8] L. Szabados, Class. Quantum Grav. 11 (1994) 1847.
[9] G. Bergqvist, Class. Quantum Grav. 9 (1992) 1753 and references therein.
[10] J.H. Yoon, J. Korean Phys. Soc. 34 (1999) 108.
[11] L. Tamburino and J. Winicour, Phys. Rev. 150 (1966) 1039.
[12] R. d'Inverno and J. Stachel, J. Math. Phys. 19 (1978) 2447.
[13] R. d'Inverno and J. Smallwood, Phys. Rev. D 22 (1980) 1233.
[14] J.H. Yoon, Phys. Lett. B 451 (1999) 296.
[15] J.H. Yoon, Class. Quantum Grav. 16 (1999) 1863.
[16] E.T. Newman and T. Unti, J. Math. Phys. 3 (1962) 891.
[17] E.T. Newman and K.P. Tod, *Asymptotically Flat Space-Times in General Relativity and Gravitation*, Vol.2, ed. A. Held (Plenum Press 1980).
[18] B. Carter, Comm. Math. Phys. 10 (1968) 280.
[19] L. Hughston, R. Penrose, P. Sommers, and M. Walker, Comm. Math. Phys. 27 (1972) 303.
[20] B. Carter and R.G. McLenaghan, Phys. Rev. D 19 (1979) 1093.
[21] H. Bondi, M. van der Burg, and A. Metzner, Proc. R. Soc. Lond. A 269 (1962) 20.
[22] C.R. Prior, Proc. R. Soc. Lond. A 354 (1977) 379.
[23] R. Wald, *General Relativity* (The University of Chicago Press 1984).
[24] R. Penrose and W. Rindler, *Spinors and space-time*, vol. 2 (Cambrige University Press 1986).
[25] P.T. Chruściel, J. Jezierski, and J. Kijowski, *Hamiltonian field theory in the radiating regime* (Springer 2002).
[26] A. Ashtekar and M. Streubel, Proc. R. Soc. Lond. A 376 (1981) 585.
[27] R. Price and K. Thorne, Phys. Rev. D 33 (1986) 915.
[28] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press 1989).